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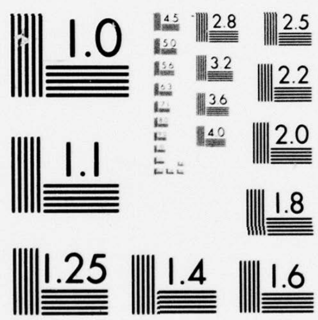

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THE AUTOCOVARIANCE FUNCTION DETERMINED VIA THE Z-TRANSFORM, WITH APPLICATION TO BOX JENKINS FORECASTING MODELS*

Research Report No. 78-5

by

Eginhard J. Muth

LEVEL II

May, 1978

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*This report supersedes Research Report No. 76-20 — A032 103

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This research was supported in part by the Army Research Office, Triangle Park, NC, under contract number DAHC04-75-G-0150.

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER ~~78-5~~ 78-5 (14) RR-78-5	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER (9)	
4. TITLE (and Subtitle) (6) The Autocovariance Function Determined Via the Z-Transform, with Application to Box Jenkins Forecasting Models.	5. TYPE OF REPORT & PERIOD COVERED Research rept., ~~Technical~~
	6. PERFORMING ORG. REPORT NUMBER ~~78-5~~ 78-5
7. AUTHOR(s) (10) Eginhard J. Muth	8. CONTRACT OR GRANT NUMBER(s) (15) DAHC04-75-G-0150
9. PERFORMING ORGANIZATION NAME AND ADDRESS Industrial and Systems Engineering University of Florida Gainesville, FL 32611	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (16) 20061102A14D (17)
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Research Office, P.O. Box 12211 Triangle Park, NC 27709	12. REPORT DATE (11) May, 1978
	13. NUMBER OF PAGES 40 (12) 41 P.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

N/A

18. SUPPLEMENTARY NOTES

(18) ARO

(19) 12640.28-M

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Box-Jenkins method
forecasting
stochastic processes
linear filters

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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ABSTRACT

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1. INTRODUCTION

An important element in the development of forecasting models is the autocovariance function of the process to be modeled. One obtains information about the nature of this function by computing the sample autocorrelation function from a sequence of observed values of the process, and one then uses this information to establish the structure of an appropriate model. The method of Box and Jenkins assumes that the process is generated as the output of a linear filter whose input is white noise. Thus the following question arises. Given the autoregressive-moving average (ARMA) equation that describes a particular filter, what is the autocovariance function of the output of that filter and how are the properties of that function influenced through the choice of filter parameters. The autocovariance function can be computed recursively in accordance with the method given in Box and Jenkins [1], but that method does not, a priori, shed much light on the nature of the autocovariance function.

In this paper we develop a method for obtaining a closed form expression for the autocovariance function of a discrete-time stationary stochastic process. The method is based on concepts from systems engineering and makes use of the Z-transform. It permits one to express the autocovariance function in terms of the parameters of the ARMA equation, without the need for solving systems of equations as would be required by the classical method. As a consequence we give expressions for ARMA (p, q) with $p > 1$ and $q > 1$, which are not to be found in Box and Jenkins [1]. Described briefly the method involves first expressing the Z-transform of the autocovariance function, then performing a partial fraction expansion, and then inverting the expansion.

The paper is organized as follows. In Section 2 we discuss the filter impulse response and the filter transfer function and their relationship to

the ARMA equation. The autocovariance function is defined in Section 3 and is derived from the impulse response. It is also indicated how the autocovariance function may be obtained from the ARMA equations by the classical method. Section 4 develops, on the conceptual level, the method for the inversion of the transform of the autocovariance function. The practical implementation of the method of partial fraction expansion for linear factors, repeated linear factors, and quadratic factors is shown in Section 5. Finally, a summary of solved cases is given in Section 6.

2. IMPULSE RESPONSE AND TRANSFER FUNCTION

We consider a discrete-time stationary stochastic process x_n which is generated as the output of a stable linear filter whose input is the stationary process a_n. In the terminology and notation of Box and Jenkins [1], x_n is an ARMA (p, q) process, if we have the input-output relation

$$x_n - \phi_1 x_{n-1} - \cdots - \phi_p x_{n-p} = a_n - \theta_1 a_{n-1} - \cdots - \theta_q a_{n-q} \tag{1}$$

The process x_n can also be represented in terms of the *impulse response* ψ_n of the filter, namely

$$x_n = \psi_0 a_n + \psi_1 a_{n-1} + \psi_2 a_{n-2} + \cdots \tag{2}$$

or

$$x_n = \psi_n * a_n \tag{3}$$

where the symbol * denotes convolution. If the coefficients of a_n and x_n are 1, as in (1), then we always have $\psi_0 = 1$. Also, the impulse response is 0 for $n < 0$.

Now we define the bilateral Z-transform of x_n as

$$\bar{x}(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n} \tag{4}$$

and we let $\bar{a}(z)$ and $\bar{\psi}(z)$ be defined analogously. We remark that for sequences which are zero for negative values of n, such as is the case with ψ_n, the bilateral Z-transform is the same as the ordinary (unilateral) Z-transform.

Equation (3), when transformed in accordance with (4), and with the convolution rule, becomes

$$\bar{x}(z) = \bar{\psi}(z)\bar{a}(z) \tag{5}$$

The shifting rule for Z-transforms states that the transform of x_{n-k} is $z^{-k}\bar{x}(z)$. Transforming equation (1) according to this rule and arranging terms yields

$$\bar{x}(z) = \frac{1 - \theta_1 z^{-1} - \cdots - \theta_q z^{-q}}{1 - \phi_1 z^{-1} - \cdots - \phi_p z^{-p}} \bar{a}(z) \tag{6}$$

Comparison of (6) with (5) shows that

$$\bar{\psi}(z) = \frac{1 - \theta_1 z^{-1} - \cdots - \theta_q z^{-q}}{1 - \phi_1 z^{-1} - \cdots - \phi_p z^{-p}} \tag{7}$$

The function $\bar{\psi}(z)$ is known as the *transfer function* of the filter. It is customary to express $\bar{\psi}(z)$ as a rational function of z, that is, as the ratio of two polynomials. This becomes necessary if we wish to carry out a *partial fraction expansion.* We multiply both the numerator and the denominator of (7) by z^m, where $m = \max(p, q)$. The result is

$$\bar{\psi}(z) = \frac{z^m - \theta_1 z^{m-1} - \cdots - \theta_q z^{m-q}}{z^m - \phi_1 z^{m-1} - \cdots - \phi_p z^{m-p}} = \frac{N(z)}{D(z)} \tag{8}$$

where both N(z) and D(z) are polynomials of degree m in z. D(z) is the *characteristic polynomial* of the difference equation (1). The *zeros* of D(z) are the *poles* of the transfer function. The poles of $\psi(z)$ play an important role since their location in the complex plane determines the nature of the impulse response. A simple pole at $z = \alpha$ gives rise to a geometric sequence in n, according to the transform pair

$$\frac{Az}{z-\alpha} \longleftrightarrow A\alpha^n u_n \tag{9}$$

where the symbol <—> denotes "transforms into", and where u_n is the unit-step sequence defined as

$$u_n = \begin{cases} 1 \text{ for } n \geq 0 \\ 0 \text{ otherwise} . \end{cases} \tag{10}$$

A pole of multiplicity r at $z = \alpha$ gives rise to a geometric sequence multiplied by a polynomial, according to the transform pair

$$A\left(\frac{z}{z-\alpha}\right)^r \longleftrightarrow A(n+r-1)(n+r-2) \cdots (n+1)\alpha^n u_n \tag{11}$$

A pole, simple or repeated, at $z = 0$ gives rise to an impulse according to the transform pair

$$\frac{A}{z^r} \longleftrightarrow A\delta_{n-r} \tag{12}$$

where the unit-impulse sequence (or delta sequence) is defined as

$$\delta_{n-r} = \begin{cases} 1 \text{ for } n = r \\ 0 \text{ otherwise} \end{cases} \tag{13}$$

A pair of complex conjugate poles with absolute value α and argument $\pm\,\beta$ has the associated quadratic factor

$$D_q(z) = (z - \alpha e^{i\beta})(z - \alpha e^{-i\beta}) = z^2 - 2\alpha z \cos\beta + \alpha^2; \quad i = \sqrt{-1} \tag{14}$$

and gives rise to a sinusoidal sequence with geometric amplitude according to the transform pair

$$\frac{A[z^2 \sin\gamma + \alpha z \sin(\beta-\gamma)]}{z^2 - 2\alpha z \cos\beta + \alpha^2} \longleftrightarrow A\alpha^n \sin(n\beta + \gamma)u_n \tag{15}$$

Only stable filters are admissible for our consideration since *stability* of the filter is necessary in order for the output process x_n to be stationary, and thus necessary for the existence of the autocovariance function. Stability

implies that $\psi_n \to 0$ as $n \to \infty$, or equivalently, that all poles of the transfer function lie inside the unit circle. The transfer function $\bar{\psi}(z)$ has exactly m poles, which we denote $\alpha_1, \alpha_2, \ldots, \alpha_p$, for $m = p$. If $q > p$ then $m = q$, and $q - p$ of these poles are at $z = 0$. We thus have in factored form

$$D(z) = \begin{cases} (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_p) & \text{if } p \geq q \\ (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_p)\, z^{p-q} & \text{if } q > p \end{cases} \tag{16}$$

and we require for stability that $|\alpha_i| < 1$ for all i. Furthermore, the transform $\bar{\psi}(z)$ exists (converges) for values of z outside a circle of radius R, where

$$R = \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_p|\} \tag{17}$$

is the radius of convergence of $\bar{\psi}(z)$. For $|z| \leq R$, $\bar{\psi}(z)$ is not defined.

The method of partial fraction expansion allows us to decompose the transfer function (8) into a sum of recognizable elements of the form of the left sides of (9), (11), (12), (15), which have simple inverse transforms. Details of the method will be found in [2], Chapter 6. A unique decomposition is possible only for proper rational functions. For this reason we first divide $\bar{\psi}(z)$ by z. The expansion then has the general form

$$\frac{\bar{\psi}(z)}{z} = \frac{N(z)}{zD(z)} = \sum_i \frac{N_i(z)}{D_i(z)} \tag{18}$$

where the $D_i(z)$ are the factors of D(z), which may be simple linear factors, or repeated linear factors, or quadratic factors. In addition one of the $D_i(z)$ is the factor z if N(z) is not divisible by z. In (18) each term $N_i(z)/D_i(z)$ is itself a proper rational function. Therefore the degree of the polynomial $N_i(z)$ is 0 for simple linear factors, $r - 1$ for repeated linear factors of multiplicity r, and 1 for quadratic factors. Detailed examples of the partial fraction expansion of $\bar{\psi}(z)$ are given in Section 5.

3. THE AUTOCOVARIANCE FUNCTION

We denote by μ_k and γ_k the *autocovariance functions* of the processes a_n and x_n. These are functions of the lag k and defined as

$$\mu_k = \text{Cov}[a_n, a_{n+k}] \tag{19}$$

$$\gamma_k = \text{Cov}[x_n, x_{n+k}] \tag{20}$$

It is a fundamental result that γ_k can be represented as the output of a linear filter with impulse response g_k whose input is μ_k, see Papoulis [3]. Thus we have

$$\gamma_k = g_k * \mu_k \tag{21}$$

and in the transform domain

$$\bar{\gamma}(z) = \bar{g}(z)\bar{\mu}(z) \tag{22}$$

The significant part of the result is that g_k is obtained from ψ_k as

$$g_k = \psi_k * \psi_{-k} \tag{23}$$

and that the transform of this relation is

$$\bar{g}(z) = \bar{\psi}(z)\bar{\psi}(\tfrac{1}{z}) \tag{24}$$

where $\bar{\psi}(\frac{1}{z})$ is the transform of ψ_{-k}. We note that g_k is a *noncausal* impulse response because it is nonzero for $k < 0$, which means that the filter output precedes its cause (the input). Since $\bar{\psi}(z)$ exists for $|z| > R$, where $R < 1$, it follows that $\bar{\psi}(\frac{1}{z})$ exists for $|z| < 1/R$. Hence the function $\bar{g}(z)$ exists in the annulus of convergence $R < |z| < 1/R$. We have from (24) that

$$\bar{g}(\tfrac{1}{z}) = \bar{g}(z) \tag{25}$$

which implies that g_k is a symmetric sequence. This fact is also seen from (23) since

$$g_{-k} = \psi_{-k} * \psi_k = g_k \tag{26}$$

If the input process a_n is *white*, which means that it is *uncorrelated*, then

$$\mu_k = \sigma_a^2 \delta_k \tag{27}$$

where $\sigma_a^2 = \mathrm{Var}[a_n]$, and (21) and (22) become

$$\gamma_k = \sigma_a^2 g_k \tag{28}$$

$$\bar{\gamma}(z) = \sigma_a^2 \bar{g}(z) \tag{29}$$

The problem now at hand is to determine the sequence g_k. This can be done using the relation (23) which is explicitly

$$g_k = \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \tag{30}$$

but this procedure does not generally yield closed form expressions for g_k. Our approach therefore is to invert the bilateral transform $\bar{g}(z)$, which Box and Jenkins call a *covariance generating function*. The details of the inversion will be developed in the next section. Prior to that we briefly discuss another method for finding γ_k in order to allow a comparison with our method.

This method for computing γ_k is alluded to in [1], where an example is given for the ARMA (1, 1) process. If one multiplies equation (1) by a_{n-j}, $j = 0, 1, \ldots, q$, and takes expected values, the result is the system of $q + 1$ equations

$$\begin{aligned}
&\omega_0 = \sigma_a^2 \\
&\omega_1 - \phi_1\omega_0 = - \theta_1\sigma_a^2 \\
&- \\
&\omega_j - \phi_1\omega_{j-1} - \cdots - \phi_j\omega_0 = - \theta_j\sigma_a^2; \quad j = 2, 3, \ldots, q \\
&\qquad\qquad\qquad \phi_j = 0 \quad \text{if } j > p
\end{aligned} \tag{31}$$

where $\omega_j = \mathrm{Cov}[x_n, a_{n-j}]$ is the *crosscovariance* of the processes x_n and a_n. It is apparent that this system of equations can be solved recursively for $\omega_0, \omega_1, \ldots, \omega_q$. Next one multiplies equation (1) by x_{n-k}, $k = 0, 1, \ldots$ and takes expected values. This gives rise first to the system of $p + 1$

simultaneous equations defining $\gamma_0, \gamma_1, \ldots, \gamma_p$

$$\gamma_0 - \phi_1\gamma_1 - \cdots - \phi_p\gamma_p = \omega_0 - \theta_1\omega_1 - \cdots - \theta_q\omega_q$$

$$\gamma_1 - \phi_1\gamma_0 - \cdots - \phi_p\gamma_{p-1} = - \theta_1\omega_0 - \cdots - \theta_q\omega_{q-1}$$

$$- -$$

$$\gamma_p - \phi_1\gamma_{p-1} - \cdots - \phi_p\gamma_p = - \theta_p\omega_0 - \cdots - \theta_q\omega_{q-p} \tag{32}$$

where $\theta_j = 0$ if $j > q$. Thus, if $q < p$ the last $p - q$ equations of (32) have a right hand side equal to zero. It gives rise secondly, if $q > p$, to the following $q - p$ equations

$$\gamma_{p+j} - \phi_1\gamma_{p+j-1} - \cdots - \phi_p\gamma_j = - \theta_{p+j}\omega_0 - \cdots - \theta_q\omega_{q-p-j} \tag{33}$$

$$j = 1, 2, \ldots, q-p$$

and thirdly to the infinite system of equations

$$\gamma_{r+j} - \phi_1\gamma_{r+j-1} - \cdots - \phi_p\gamma_{r+j-p} = 0; \qquad j = 1, 2, \ldots \tag{34}$$

where $r = \max\{p, q\}$. Equations (33) and (34) constitute jointly a nonhomogeneous difference equation which defines γ_k for $k \geq 1$. Equation (34) constitutes a homogeneous difference equation which defines γ_k for $k \geq 1 + \max\{0, q-p\}$. In order to compute γ_k one needs to solve equations (31) through (34). This presents no problem in the case where numerical values are given for the parameters ϕ_i and θ_j, and where (34) is solved recursively. The situation is different, however, if one needs a closed form solution expressed in terms of the parameters ϕ_i and θ_j. In this case one first solves (31) recursively and expresses $\omega_1, \omega_2, \ldots, \omega_q$ in terms of the ϕ_i and θ_j. These expressions are substituted in (32) and (33). Next one solves (32). Since these are simultaneous equations, the solution requires essentially the inversion of a matrix of size $p + 1$ in symbolic form, and this may be a difficult part. Equation (33) can again be solved recursively, but is needed only if $q > p$. The homogeneous

equation (34) may be solved by the classical method, yielding the expression

$$\gamma_k = A_1\alpha_1^k + A_2\alpha_2^k + \cdots + A_p\alpha_p^k \tag{35}$$

where the α_i are the roots of the equation

$$z^p - \phi_1 z^{p-1} - \cdots - \phi_p = 0 \tag{36}$$

The coefficients A_i in (35) are then determined by solving p simultaneous equations consisting of (35) with $k = r-1, r-2, \ldots, r-p$, where $r = \max\{p, q\}$ and where the values of γ_k are substituted from the solution of (32) and (33). Here again the solution of simultaneous equations in symbolic form may prove difficult. It should be apparent that this procedure for finding a closed form expression for γ_k can be a messy one. On the other hand, it is well known that transform methods serve to simplify the solution of difference equations. This is especially true of the method presented in the following section. The reason is that this method does not require a separate solution of the systems (31) and (32), which are not part of the difference equation, and moreover it does not involve any solution of simultaneous equations at all.

We remark as an aside that the *power spectral density* function (or power spectrum) of the process x_n is readily determined from the transfer function $\bar{\psi}(z)$. The power spectrum $S(\omega)$ is the Fourier transform of the autocovariance function γ_n. The Fourier transform is obtained from the bilateral Z-transform by setting $z = e^{i\omega}$, $i = \sqrt{-1}$. Thus we have with (29) and (24)

$$S(\omega) = \bar{\gamma}(e^{i\omega}) = \sigma_a^2\, \bar{\psi}(e^{i\omega})\bar{\psi}(e^{-i\omega}) = \sigma_a^2|\bar{\psi}(e^{i\omega})|^2 \tag{38}$$

This result states that the absolute value of the transfer function must be determined for values of z on the unit circle. A graphical method for quickly assessing the form of $|\psi(e^{i\omega})|$ from the zeros and poles of the transfer function is given in [2], pages 321 to 323.

4. THE INVERSION OF $\bar{g}(z)$

In this section we develop the method for deriving a closed form expression for g_k from our knowledge of the ARMA equation. Specific examples will then be computed in the next section. The transfer function $\bar{\psi}(z)$ is readily determined from the ARMA equation, see (8). Next, we get the transform of g_k from (24). Thus we must now address the question of inverting $\bar{g}(z)$. It is possible to have different sequences with the same bilateral transform. For example, the sequences

$$b_n = \begin{cases} \rho^n & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

and

$$c_n = \begin{cases} -\rho^{n-1} & \text{for } n < 0 \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

have the same tranform $\bar{b}(z) = \bar{c}(z) = z/(z-\rho)$. The difference is that $\bar{b}(z)$ exists for $|z| > \rho$ whereas $\bar{c}(z)$ exists for $|z| < \rho$. Therefore, to determine uniquely the inverse of a bilateral transform, one must know the annulus of convergence of that transform. We had shown before that the annulus of convergence of $\bar{g}(z)$ is $R < |z| < 1/R$, with R given by (17).

The function $\bar{g}(z)$ is itself a rational function. Replacing z by 1/z in (7) we obtain

$$\bar{\psi}(\tfrac{1}{z}) = \frac{1 - \theta_1 z - \cdots - \theta_q z^q}{1 - \phi_1 z - \cdots - \phi_p z^p} \tag{41}$$

Since each pole of $\bar{\psi}(z)$ at $z = \alpha_i$, where $\alpha_i \neq 0$, gives rise to a pole of $\bar{\psi}(1/z)$ at $1/z = \alpha_i$, or $z = 1/\alpha_i$, we can write (41) as

$$\bar{\psi}(\tfrac{1}{z}) = \frac{P(z)}{Q(z)} \tag{42}$$

where

$$Q(z) = (z - 1/\alpha_1)(z - 1/\alpha_2) \cdots (z - 1/\alpha_p) \tag{43}$$

In the case $q > p$, $\bar{\psi}(z)$ has poles also at $z = 0$, these become poles of $\bar{\psi}(1/z)$ at $z = \infty$, which means that the degree of $P(z)$ is greater than the degree of $Q(z)$. Multiplication of (42) by (8) yields the rational function

$$\bar{g}(z) = \frac{N(z)P(z)}{D(z)Q(z)} \tag{44}$$

where the numerator has degree $m + q$ and the denominator has degree $m + p$.

In order to facilitate the inversion of (44) we will exploit the symmetry property of g_k and we will reduce the inversion of a bilateral Z-transform to the simpler inversion of an ordinary Z-tranform. Since g_k is symmetric we can represent it by its value at $k = 0$ and by the symmetric right and left lobes. Thus, if we define

$$h_k = \begin{cases} 0 \text{ for } k \leq 0 \\ g_k \text{ for } k > 0 \end{cases} \tag{45}$$

we can write

$$g_k = g_0 \delta_k + h_k + h_{-k} \tag{46}$$

The tranform of (46) is

$$\bar{g}(z) = g_0 + \bar{h}(z) + \bar{h}(\tfrac{1}{z}) \tag{47}$$

where $\bar{h}(z)$ is an ordinary Z-tranform which converges for $|z| > R$ and where $\bar{h}(\frac{1}{z})$ converges for $|z| < 1/R$. From (45) and the definition of the Z-transform follows the asymptotic behavior

$$\bar{h}(z) \sim \frac{g_1}{z} \qquad \text{as} \quad z \to \infty \tag{48}$$

$$\bar{h}(\tfrac{1}{z}) \sim g_1 z \qquad \text{as} \quad z \to 0 \tag{49}$$

where condition (49) implies that $\bar{h}(\frac{1}{z})$ is divisible by z. We note the following facts. First, $\bar{h}(z)$ and $\bar{h}(\frac{1}{z})$ are rational functions. If $\bar{h}(z)$ were not rational then $\bar{h}(\frac{1}{z})$ would not be rational and $\bar{g}(z)$ could not be rational, which is a contradiction. Second, the poles of $\bar{h}(z)$ and $\bar{h}(\frac{1}{z})$ combined must be the poles of $\bar{g}(z)$, that is, the poles of $\bar{\psi}(z)$ and $\bar{\psi}(\frac{1}{z})$ combined. Third, for each pole that $\bar{h}(z)$ has at $z = \alpha_i \neq 0$, $\bar{h}(\frac{1}{z})$ must have a pole at $z = 1/\alpha_i$. Since $\bar{h}(z)$ converges for $|z| > R$ all of its poles lie inside or on the circle of radius R, and hence are identical to the poles of $\bar{\psi}(z)$. Likewise, the poles of $\bar{h}(\frac{1}{z})$ are identical to the poles of $\bar{\psi}(\frac{1}{z})$. We can therefore write (47) as

$$\bar{g}(z) = g_0 + \frac{N_1(z)}{D(z)} + \frac{N_2(z)}{Q(z)} \tag{50}$$

where $N_1(z)$ is a polynomial of degree m - 1 and $N_2(z)$ is a polynomial of degree q. Because of the symmetry of g_k we have complete information about it if we know g_0 and h_k. We let

$$f_k = g_0\delta_k + h_k \tag{51}$$

and

$$\bar{f}(z) = g_0 + \bar{h}(z) \tag{52}$$

Clearly, $\bar{f}(z)$ is an ordinary Z-transform which can be written as

$$\bar{f}(z) = \frac{M(z)}{D(z)} \tag{53}$$

where D(z) is the characteristic polynomial given by (16), and where the degree of M(z) is the same as the degree of D(z) since

$$\bar{f}(z) \longrightarrow g_0 \quad \text{as} \quad z \to \infty \tag{54}$$

Equation (50) becomes

$$\bar{g}(z) = \frac{M(z)}{D(z)} + \frac{N_2(z)}{Q(z)} \tag{55}$$

Our objective is to determine $\bar{f}(z)$ from $\bar{g}(z)$. This means that we must find the coefficients of the polynomial $M(z)$, which is most expediently done by partial fraction expansion. However, a unique expansion of $M(z)/D(z)$ is not possible because it is not a proper rational function. We therefore divide $\bar{g}(z)$ by z and obtain

$$\frac{\bar{g}(z)}{z} = \frac{M(z)}{zD(z)} + \frac{N_3(z)}{Q(z)} \tag{56}$$

Note that because of (49) $N_2(z)$ is divisible by z, hence we have $N_3(z) = N_2(z)/z$. Now we can write the expansion

$$\frac{\bar{f}(z)}{z} = \frac{M(z)}{zD(z)} = \sum_i \frac{M_i(z)}{D_i(z)} \tag{57}$$

which is analogous to (18). Furthermore

$$\frac{\bar{g}(z)}{z} = \frac{N(z)P(z)}{zD(z)Q(z)} = \frac{N_3(z)}{Q(z)} + \sum_i \frac{M_i(z)}{D_i(z)} \tag{58}$$

The $D_i(z)$ are the factors of $D(z)$ and include an additional factor z if $N(z)$ is not divisible by z. The $M_i(z)$ are polynomials that have one degree less than $D_i(z)$. We see that the sequence f_k has exactly the same structure and is made up of the same sequences as the impulse response ψ_k, only with different coefficients. The final and important result is that the coefficients of the $M_i(z)$ in (58) are determined from $\bar{g}(z)/z$ by the identical procedure by which the coefficients of the $N_i(z)$ in (18) are determined from $\bar{\psi}(z)/z$. Although (58) in comparison with (18) has the additional term $N_3(z)/Q(z)$, this term does not affect the procedure (which is based solely on the factors $D_i(z)$) because $Q(z)$ does not contain any of the factors $D_i(z)$.

The analysis up to this point has focused on the underlying concepts. We emphasize however that the implementation of the method is very simple. It involves the following steps.

1. Determine $\bar{g}(z)$ from the ARMA equation.

2. Perform a partial fraction expansion of $\bar{g}(z)/z$ into terms whose denominators comprise all factors $D_i(z)$.

3. Multiply the expansion by z and invert.

Specific examples of this procedure for simple linear factors, repeated linear factors, and quadratic factors are given in the next section. A summary of the results of a number of solved cases is given in Section 6. Details of their derivation are available from the author upon request.

5. EXAMPLE CASES

5.1 SIMPLE LINEAR FACTORS

Example 1: We consider the case ARMA (2, 2) for which (1) becomes

$$x_n - \phi_1 x_{n-1} - \phi_2 x_{n-2} = a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2} \tag{59}$$

The transfer function (8) is

$$\bar{\psi}(z) = \frac{z^2 - \theta_1 z - \theta_2}{z^2 - \phi_1 z - \phi_2} = \frac{N(z)}{D(z)} \tag{60}$$

Now let the zeroes of D(z) be α_1 and α_2. We then have

$$D(z) = (z - \alpha_1)(z - \alpha_2) \tag{61}$$

and $\phi_1 = \alpha_1 + \alpha_2$, $\phi_2 = -\alpha_1\alpha_2$. We first find the transfer function. Division of $\bar{\psi}(z)$ by z introduces an additional pole at $\alpha_0 = 0$ because N(z) is not divisible by z. The partial fraction expansion is

$$\frac{\bar{\psi}(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-\alpha_1} + \frac{A_2}{z-\alpha_2} \tag{62}$$

where we find the coefficients by the standard method, see [2]

$$A_i = \left.\frac{(z-\alpha_i)\bar{\psi}(z)}{z}\right|_{z=\alpha_i}, \qquad i = 0, 1, 2 \tag{63}$$

Specifically we obtain

$$A_0 = \left.\frac{z^2 - \theta_1 z - \theta_2}{(z-\alpha_1)(z-\alpha_2)}\right|_{z=0} = -\frac{\theta_2}{\alpha_1\alpha_2} \tag{64}$$

And in the same manner we find

$$A_1 = \frac{\alpha_1^2 - \alpha_1\theta_1 - \theta_2}{\alpha_1(\alpha_1-\alpha_2)} \tag{65}$$

$$A_2 = \frac{\alpha_2^2 - \alpha_2\theta_1 - \theta_2}{\alpha_2(\alpha_2-\alpha_1)} \tag{66}$$

Multiplication of (62) by z and inversion yields the impulse response

$$\psi_n = A_0\delta_n + A_1\alpha_1^n + A_2\alpha_2^n \tag{67}$$

To obtain the sequence f_k we note that it must have the same structure as the impulse response, namely

$$f_k = B_0\delta_k + B_1\alpha_1^k + B_2\alpha_2^k \tag{68}$$

where

$$B_i = \left.\frac{(z-\alpha_i)\bar{\psi}(z)\bar{\psi}(\frac{1}{z})}{z}\right|_{z=\alpha_i}, \quad i = 0,\ 1,\ 2 \tag{69}$$

Clearly then we have

$$B_i = \bar{\psi}(\frac{1}{\alpha_i})A_i \tag{70}$$

We obtain from (60)

$$\bar{\psi}(\frac{1}{z}) = \frac{1 - \theta_1 z - \theta_2 z^2}{(1-\alpha_1 z)(1-\alpha_2 z)} \tag{71}$$

and the coefficients are from (70) and (71)

$$B_0 = A_0 \tag{72}$$

$$B_1 = \frac{1 - \alpha_1\theta_1 - \alpha_1^2\theta_2}{(1-\alpha_1^2)(1-\alpha_1\alpha_2)}\ A_1 \tag{73}$$

$$B_2 = \frac{1 - \alpha_2\theta_1 - \alpha_2^2\theta_2}{(1-\alpha_1\alpha_2)(1-\alpha_2^2)} A_2 \tag{74}$$

This example demonstrates amply the simplicity of the procedure and its substantial advantages over the approach mentioned in Section 3. We note that it is not necessary to first find the impulse response and its coefficients A_i. The structure of (68) is known from the poles of $\bar{\psi}(z)/z$ and the coefficients B_i may be obtained directly from (69). However, since these coefficients will always satisfy the product form (70), it seems convenient to express them in that form.

Example 2: Whenever the order p of the autoregressive part is greater than the order q of the moving average part, the division of $\bar{\psi}(z)$ by z merely cancels an existing factor z in the numerator N(z). As an example we take the system ARMA (2, 1). Equation (1) becomes

$$a_n - \phi_1 x_{n-1} - \phi_2 x_{n-2} = a_n - \theta_1 a_{n-1} \tag{75}$$

and the transfer function is

$$\bar{\psi}(z) = \frac{z^2 - \theta_1 z}{z^2 - \phi_1 z - \phi_2} \tag{76}$$

Now let the denominator of $\bar{\psi}(z)$ be factored as

$$D(Z) = (z - \alpha)(z - \beta) \tag{77}$$

we have

$$\frac{\bar{\psi}(z)}{z} = \frac{z - \theta_1}{(z - \alpha)(z - \beta)} \tag{78}$$

and the expansion (41) becomes

$$\frac{\bar{\psi}(z)}{z} = \frac{A_1}{z - \alpha} + \frac{A_2}{z - \beta} \tag{79}$$

The impulse response is

$$\psi_n = A_1\alpha^n + A_2\beta^n \tag{80}$$

where, from (63)

$$A_1 = \frac{\alpha - \theta_1}{\alpha - \beta} ; \qquad A_2 = \frac{\beta - \theta_1}{\beta - \alpha} \tag{81}$$

Furthermore

$$f_k = B_1\alpha^k + B_2\beta^k \tag{82}$$

where

$$B_1 = \frac{\frac{1}{\alpha^2} - \theta_1 \frac{1}{\alpha}}{(\frac{1}{\alpha} - \alpha)(\frac{1}{\alpha} - \beta)} A_1 = \frac{1 - \alpha\theta_1}{(1 - \alpha^2)(1 - \alpha\beta)} A_1 \tag{83}$$

$$B_2 = \frac{1 - \beta\theta_1}{(1 - \alpha\beta)(1 - \beta^2)} A_2 \tag{84}$$

5.2 REPEATED LINEAR FACTORS

We consider now the case where one of the factors of $zD(z)$ occurs more than once. Let β be a zero of $zD(z)$ with multiplicity k. Then we can write

$$zD(z) = (z - \beta)^k D_2(z) \tag{85}$$

and the transfer function is expanded as

$$\frac{\bar{\psi}(z)}{z} = \frac{N(z)}{zD(z)} = \frac{N_1(z)}{(z - \beta)^k} + \frac{N_2(z)}{D_2(z)} \tag{86}$$

where the term of interest has the further expansion

$$\frac{N_1(z)}{(z - \beta)^k} = \frac{A_0}{(z - \beta)^k} + \frac{A_1}{(z - \beta)^{k-1}} + \cdots + \frac{A_{k-1}}{z - \beta} \tag{87}$$

We need to determine the coefficients $A_1, A_2, \ldots, A_{k-1}$. To this end we define the function

$$P(z) = \frac{\bar{\psi}(z)}{z} (z - \beta)^k \tag{88}$$

It now follows from (86) and (87) that

$$P(z) = A_0 + A_1(z - \beta) + \cdots + A_{k-1}(z - \beta)^{k-1} + \frac{N_2(z)}{D_2(z)}(z - \beta)^k \quad (89)$$

and the required coefficients are isolated by evaluating P(z) and its derivatives at $z = \beta$. Thus we have

$$A_0 = P(\beta)$$

$$A_1 = \frac{d}{dz} P(z)\Big|_{z=\beta}$$

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$$A_j = \frac{1}{j!} \frac{d^j}{dz^j} P(z)\Big|_{z=\beta} \quad (90)$$

After the coefficients have been determined we multiply (87) by z and invert. The inverse transformation is denoted by the operator symbol $\underset{\sim}{Z}^{-1}$. In the case $\beta \neq 0$ we use the rule

$$\underset{\sim}{Z}^{-1}\left[\frac{z}{(z - \beta)^j}\right] = \binom{n}{j - 1} \beta^{n+1-j} \quad (91)$$

from which it follows with (96) that

$$\underset{\sim}{Z}^{-1}\left[\frac{N_1(z)}{(z - \beta)^k}\right] = \sum_{j=0}^{k-1} A_j \binom{n}{k - j - 1} \beta^{n+1-k+j} \quad (92)$$

In the case $\beta = 0$ we require the inverse transform of z^{1-j} which is simply a unit impulse at the position $n = j - 1$, namely

$$\underset{\sim}{Z}^{-1}\left[\frac{z}{z^j}\right] = \delta_{n+1-j} \quad (93)$$

and the inverse transform of (87) becomes

$$\underset{\sim}{Z}^{-1}\left[\frac{N_1(z)}{z^k}\right] = \sum_{j=0}^{k-1} A_j \delta_{n+1-k+j} \quad (94)$$

The expansion of the transform of the function f_k in analogy to (86), is

$$\frac{\bar{f}(z)}{z} = \frac{M_1(z)}{(z - \beta)^k} + \frac{M_2(z)}{D_2(z)} \tag{95}$$

where

$$\frac{M_1(z)}{(z - \beta)^k} = \frac{B_0}{(z - \beta)^k} + \frac{B_1}{(z - \beta)^{k-1}} + \cdots + \frac{B_{k-1}}{z - \beta} \tag{96}$$

After defining the function

$$Q(z) = \frac{\bar{\psi}(z)\bar{\psi}(\frac{1}{z})}{z} (z - \beta)^k = P(z)\bar{\psi}(\frac{1}{z}) \tag{97}$$

we obtain

$$B_0 = Q(\beta) = A_0\bar{\psi}(\frac{1}{\beta})$$

$$B_1 = \frac{d}{dz} Q(z)\Big|_{z=\beta} = A_1\bar{\psi}(\frac{1}{\beta}) + A_0 \frac{d}{dz} \bar{\psi}(\frac{1}{z})\Big|_{z=\beta}$$

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$$B_j = \frac{1}{j!} \frac{d^j}{dz^j} Q(z)\Big|_{z=\beta} \tag{98}$$

It then follows that

$$\underset{\sim}{Z}^{-1}\left[\frac{M_1(z)}{(z - \beta)^k}\right] = \begin{cases} \sum\limits_{j=0}^{k-1} B_j \binom{n}{k - j - 1} \beta^{n+1-k+j} & \text{for } \beta \neq 0 \\ \sum\limits_{j=0}^{k-1} B_j \delta_{n+1-k+j} & \text{for } \beta = 0 \end{cases} \tag{99}$$

Example 3: We consider the system ARMA (2, 1) whose characteristic polynomial has a double zero at $z = \beta$. Equation (1) becomes

$$x_n - 2\beta x_{n-1} + \beta^2 x_{n-2} = a_n - \theta_1 a_{n-1} \tag{100}$$

and the transfer function is

$$\bar{\psi}(z) = \frac{z^2 - \theta_1 z}{(z - \beta)^2} \tag{101}$$

In accordance with (88) we find

$$P(z) = z - \theta_1; \quad \frac{d}{dz} P(z) = 1 \tag{102}$$

and the coefficients of the impulse response are computed from P(z) by (99) as

$$A_0 = \beta - \theta_1; \quad A_1 = 1 \tag{103}$$

The inverse transform (91), for j = 1 and j = 2 is

$$\underset{\sim}{Z}^{-1}\left[\frac{z}{z - \beta}\right] = \beta^n$$

$$\underset{\sim}{Z}^{-1}\left[\frac{z}{(z - \beta)^2}\right] = n\beta^{n-1} \tag{104}$$

and the impulse response becomes

$$\psi_n = (\beta - \theta_1)n\beta^{n-1} + \beta^n \tag{105}$$

The function Q(z) defined in (97) is

$$Q(z) = (z - \theta_1) \frac{1 - \theta_1 z}{(1 - \beta z)^2} \tag{106}$$

and

$$\frac{d}{dz} Q(z) = \frac{1 - 2\theta_1\beta + \theta_1^2 + (\beta + \theta_1^2\beta - 2\theta_1)z}{(1 - \beta z)^3} \tag{107}$$

The coefficients of f_k are computed with (98) as

$$B_0 = \frac{(\beta - \theta_1)(1 - \theta_1\beta)}{(1 - \beta^2)^2} \tag{108}$$

$$B_1 = \frac{(1+\beta^2)(1+\theta_1^2) - 4\theta_1\beta}{(1-\beta^2)^3} \tag{109}$$

Finally we have

$$f_k = B_0 k\beta^{k-1} + B_1\beta^k \tag{110}$$

Example 4: We treat here the system ARMA (1, 3), which induces a triple zero at $z = 0$. Equation (1) becomes

$$x_n - \phi_1 x_{n-1} = a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2} - \theta_3 a_{n-3} \tag{111}$$

and the transfer function is

$$\bar{\psi}(z) = \frac{N(z)}{D(z)} = \frac{z^3 - \theta_1 z^2 - \theta_2 z - \theta_3}{z^2(z - \phi_1)} \tag{112}$$

In this case $D(z)$ has a double zero at $z = 0$, and the division by z introduces the third zero. The partial fraction expansion is

$$\frac{\bar{\psi}(z)}{z} = \frac{A_0}{z^3} + \frac{A_1}{z^2} + \frac{A_2}{z} + \frac{A_3}{z - \phi_1} \tag{113}$$

where the coefficient A_3 is determined by the method for linear factors, namely

$$A_3 = \left.\frac{N(z)}{z^3}\right|_{z = \phi_1} = \frac{\phi_1^3 - \theta_1\phi_1^2 - \theta_2\phi_1 - \theta_3}{\phi_1^3} \tag{114}$$

To obtain A_0, A_1 and A_2 we use the method for repeated linear factors. From (88) follows

$$P(z) = \bar{\psi}(z)z^2 = \frac{z^3 - \theta_1 z^2 - \theta_2 z - \theta_3}{z - \phi_1} \tag{115}$$

$$\frac{d}{dz}P(z) = \frac{3z^2 - 2\theta_1 z - \theta_2}{z - \phi_1} - \frac{z^3 - \theta_1 z^2 - \theta_2 z - \theta_3}{(z - \phi_1)^2} \tag{116}$$

$$\frac{1}{2}\ \frac{d^2}{dz^2} P(z) = \frac{3z - \theta_1}{z - \phi_1} - \frac{(3z^2 - 2\theta_1 - \theta_2)}{(z - \phi_1)^2} + \frac{(z^3 - \theta_1 z^2 - \theta_2 z - \theta_3)}{(z - \phi_1)^3} \quad (117)$$

Evaluation of (115), (116) and (117) at $z = 0$ yields

$$A_0 = \frac{\theta_3}{\phi_1}$$

$$A_1 = \frac{\theta_2}{\phi_1} + \frac{\theta_3}{\phi_1^2} = \frac{\theta_2\phi_1 + \theta_3}{\phi_1^2}$$

$$A_2 = \frac{\theta_1}{\phi_1} + \frac{\theta_2}{\phi_1^2} + \frac{\theta_3}{\phi_1^3} = \frac{\theta_1\phi_1^2 + \theta_2\phi_1 + \theta_3}{\phi_1^3} \quad (118)$$

The impulse response is

$$\psi_n = A_0\delta_{n-2} + A_1\delta_{n-1} + A_2\delta_n + A_3\phi_1^n \quad (119)$$

Now, because $Q(z) = P(z)\bar{\psi}(\frac{1}{z})$, the equations (98) for the coefficients B_0, B_1, and B_2 become

$$B_0 = A_0\bar{\psi}(\tfrac{1}{z}) \Big|_{z=\theta}$$

$$B_1 = \left[A_1\psi(\tfrac{1}{z}) + A_0 \frac{d}{dz}\,\bar{\psi}(\tfrac{1}{z})\right]_{z=0}$$

$$B_2 = \left[A_2\bar{\psi}(\tfrac{1}{z}) + A_1 \frac{d}{dz}\,\bar{\psi}(\tfrac{1}{z}) + \frac{1}{2} A_0 \frac{d^2}{dz^2}\,\bar{\psi}(\tfrac{1}{z})\right]_{z=0} \quad (120)$$

where

$$\bar{\psi}(\tfrac{1}{z}) = \frac{1 - \theta_1 z - \theta_2 z^2 - \theta_3 z^3}{1 - \phi_1 z}$$

$$\frac{d}{dz}\,\bar{\psi}(\tfrac{1}{z}) = \frac{-\theta_1 - 2\theta_2 z - 3\theta_3 z^2}{1 - \phi_1 z} + \frac{\phi_1(1 - \theta_1 z - \theta_2 z^2 - \theta_3 z^3)}{(1 - \phi_1 z)^2}$$

$$\frac{d^2}{dz^2}\,\bar{\psi}(\tfrac{1}{z}) = \frac{-2\theta_2 - 6\theta_3 z}{1 - \phi_1 z} - \frac{2\phi_1(\theta_1 + 2\theta_2 z + 3\theta_3 z^2)}{(1 - \phi_1 z)^2}$$

$$+ \frac{2\phi_1^2(1 - \theta_1 z - \theta_2 z^2 - \theta_3 z^3)}{(1 - \phi_1 z)^3} \quad (121)$$

Substituting these results into (120) gives

$$B_0 = A_0$$

$$B_1 = A_1 + (\phi_1 - \theta_1)A_0$$

$$B_2 = A_2 + (\phi_1 - \theta_1)A_1 + (\phi_1^2 - \phi_1\theta_1 - \theta_2)A_0 \tag{122}$$

The function f_k is

$$f_k = B_0\delta_{k-2} + B_1\delta_{k-1} + B_2\delta_k + B_3\phi_1^k \tag{123}$$

where B_3 is obtained as

$$B_3 = A_3\bar{\psi}(\frac{1}{\phi_1}) = \frac{1 - \theta_1\phi_1 - \theta_2\phi_1^2 - \theta_1^3}{1 - \phi_1^2} A_3 \tag{124}$$

5.3 QUADRATIC FACTORS

A quadratic factor with real coefficients arises as the product of two linear factors with complex conjugate zeros. Let these zeros be at $\alpha e^{i\beta}$ and $\alpha e^{-i\beta}$, where i denotes the imaginary unit. We then obtain the quadratic factor

$$D_1(z) = (z - \alpha e^{i\beta})(z - \alpha e^{-i\beta}) = z^2 - 2\alpha z \cos\beta + \alpha^2 \tag{125}$$

Now let the denominator D(z) of the transfer function contain the quadratic factor $D_1(z)$. We can write

$$zD(z) = D_1(z)D_2(z)$$

and the transfer function has the expansion

$$\frac{\bar{\psi}(z)}{z} = \frac{N_1(z)}{D_1(z)} + \frac{N_2(z)}{D_2(z)} \tag{126}$$

$N_1(z)$ is a polynomial of degree one and has two open coefficients which are to be determined by the method of partial fraction expansion. If we write $N_1(z)$ in the form

$$N_1(z) = A[z \sin \gamma + \alpha \sin(\beta - \gamma)] \tag{127}$$

$$= A_1\alpha \sin \beta + A_2(z - \alpha \cos \beta)$$

where

$$A_1 = A \cos \gamma; \quad A_2 = A \sin \gamma$$

then the inverse transform of the first term of (126) multiplied by z is

$$\underset{\sim}{Z}^{-1} \left[\frac{zN_1(z)}{D_1(z)}\right] = A\alpha^n \sin (n\beta + \gamma)$$

$$= \alpha^n[A_1 \sin n\beta + A_2 \cos n\beta] \tag{128}$$

Thus every transform with denominator $D_1(z)$ represents a damped sinusoid of amplitude $A\alpha^n$, frequency β, and phase angle γ. We have to determine the values of A and γ or of A_1 and A_2. We first define the function

$$P(z) = \frac{\bar{\psi}(z)}{z} D_1(z) \tag{129}$$

It then follows from (126) and (127) that

$$P(z) = A_1\alpha \sin \beta + A_2(z - \alpha \cos \beta) + \frac{N_2(z)D_1(z)}{D_2(z)} \tag{130}$$

The required coefficients are isolated by letting z take on the value

$$z_0 = \alpha e^{i\beta} = \alpha(\cos \beta + i \sin \beta) \tag{131}$$

This value is a zero of $D_1(z)$, hence $D_1(z_0) = 0$. We then obtain

$$P(z_0) = A_1\alpha \sin \beta + A_2(\alpha \cos \beta + i\alpha \sin \beta - \alpha \cos \beta)$$

$$= \alpha \sin \beta(A_1 + iA_2) \tag{132}$$

and, dividing by $\alpha \sin \beta$

$$\frac{P(z_0)}{\alpha \sin \beta} = A_1 + iA_2 \tag{133}$$

The values of A_1 and A_2 are now readily determined. We note that $P(z_0)$ is simply

a complex number. Denoting its real part by R and its imaginary part by I, that is, letting

$$P(z_0) = R + iI \tag{134}$$

we find

$$A = \frac{\sqrt{R^2 + I^2}}{\alpha \sin \beta} \quad ; \quad \gamma = \arc \tan \frac{I}{R} \tag{135}$$

or

$$A_1 = \frac{R}{\alpha \sin \beta} \quad ; \quad A_2 = \frac{I}{\alpha \sin \beta} \tag{136}$$

An analogous procedure applies to the transform $\bar{f}(z)$. As the counterpart to (126) we have the expansion

$$\frac{\bar{f}(z)}{z} = \frac{M_1(z)}{D_1(z)} + \frac{M_2(z)}{D_2(z)} \tag{137}$$

where

$$\begin{aligned} M_1(z) &= B[z \sin \epsilon + \alpha \sin (\beta - \epsilon)] \\ &= B_1 \alpha \sin \beta + B_2(z - \alpha \cos \beta) \end{aligned}$$

and

$$B_1 = B \cos \epsilon; \quad B_2 = B \sin \epsilon \tag{138}$$

We define the function

$$Q(z) = \frac{\bar{\psi}(z)\bar{\psi}(\frac{1}{z})}{z} D_1(z) = P(z)\bar{\psi}(\frac{1}{z}) \tag{139}$$

and obtain

$$\frac{Q(z_0)}{\alpha \sin \beta} = B_1 + iB_2 \tag{140}$$

Now let C_1 and C_2 be the real and imaginary parts of $\bar{\psi}(\frac{1}{z_0})$. It then follows that

$$\frac{Q(z_0)}{\alpha \sin \beta} = \frac{P(z_0)}{\alpha \sin \beta} \bar{\psi}(\frac{1}{z_0}) = (A_1 + iA_2)(C_1 + iC_2) \tag{141}$$

and also that

$$B_1 = A_1C_1 - A_2C_2; \quad B_2 = A_1C_2 + A_2C_1 \tag{142}$$

Example 5: Consider the system ARMA (2, 2) governed by the equation

$$x_n - 2\alpha \cos\beta \; x_{n-1} + \alpha^2 x_{n-2} = a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2} \tag{143}$$

The transfer function is

$$\bar{\psi}(z) = \frac{z^2 - \theta_1 z - \theta_2}{z^2 - 2\alpha z \cos \beta + \alpha^2} \tag{144}$$

and the partial fraction expansion of $\bar{\psi}(z)/z$ has the form

$$\frac{\bar{\psi}(z)}{z} = \frac{A_0}{z} + \frac{A_1 \alpha \sin \beta + A_2(z - \alpha \cos \beta)}{z^2 - 2\alpha z \cos \beta + \alpha^2} \tag{145}$$

By the usual method for linear factors we find

$$A_0 = \bar{\psi}(0) = \frac{-\theta_2}{\alpha^2} \tag{146}$$

Next we let in accordance with (129)

$$P(z) = \frac{z^2 - \theta_1 z - \theta_2}{z} = z - \theta_1 - \theta_2 z^{-1} \tag{147}$$

This function is now evaluated at z_0

$$\begin{aligned} P(z_0) &= \alpha e^{i\beta} - \theta_1 - \frac{\theta_2}{\alpha} e^{-i\beta} \\ &= (\alpha - \frac{\theta_2}{\alpha}) \cos \beta - \theta_1 + i(\alpha + \frac{\theta_2}{\alpha}) \sin \beta \end{aligned} \tag{148}$$

After dividing (148) by $\alpha \sin \beta$ and comparing it with (133) we see that

$$\begin{aligned} A_1 &= (1 - \frac{\theta_2}{\alpha^2}) \cot \beta - \frac{\theta_1}{\alpha \sin \beta} \\ A_2 &= (1 + \frac{\theta_2}{\alpha^2}) = 1 - A_0 \end{aligned} \tag{149}$$

The impulse response is

$$\psi_n = A_0 \delta_n + \alpha^n (A_1 \sin \beta + A_2 \cos n\beta) \tag{150}$$

The function f_k has the same expansion as (144), namely

$$f_k = B_0\delta_k + \alpha^k(B_1 \sin k\beta + B_2 \cos k\beta) \tag{151}$$

where

$$B_0 = A_0\bar{\psi}(\frac{1}{z})\Big|_{z=0} = A_0 \tag{152}$$

In order to find B_1 and B_2 we must evaluate $\psi(\frac{1}{z})$ at $z = z_0$. We have from (144) and (125)

$$\bar{\psi}(z) = \frac{z_2 - \theta_1 z - \theta_2}{(z - \alpha e^{i\beta})(z - \alpha e^{-i\beta})} \tag{153}$$

and therefore

$$\psi(\frac{1}{z}) = \frac{1 - \theta_1 z - \theta_2 z^2}{(1 - z\alpha e^{i\beta})(1 - z\alpha e^{-i\beta})} \tag{154}$$

Substitution of $z_0 = \alpha e^{i\beta}$ in (154) yields

$$\bar{\psi}(\frac{1}{z_0}) = \frac{1 - \theta_1\alpha e^{i\beta} - \theta_2\alpha^2 e^{2i\beta}}{(1 - \alpha^2 e^{2i\beta})(1 - \alpha^2)}$$

$$= \frac{(1 - \theta_1\alpha e^{i\beta} - \theta_2\alpha^2 e^{2i\beta})(1 - \alpha^2 e^{-2i\beta})}{(1 - 2\alpha^2 \cos 2\beta + \alpha^4)(1 - \alpha^2)} \tag{155}$$

We let D_1 and D_2 be the real and imaginary parts of the numerators of (155). We thus have

$$D_1 + iD_2 = 1 - \theta_1\alpha e^{i\beta} - \theta_2\alpha^2 e^{2i\beta} - \alpha^2 e^{-2i\beta} + \theta_1\alpha^3 e^{-i\beta} + \theta_2\alpha^4 \tag{156}$$

and after separating real and imaginary parts

$$D_1 = 1 + \theta_2\alpha^4 + \theta_1(\alpha^3 - \alpha) \cos \beta - \alpha^2(1 + \theta_2) \cos 2\beta \tag{157}$$

$$D_2 = -\theta_1(\alpha + \alpha^3) \sin \beta + \alpha^2(1 - \theta_2) \sin 2\beta \tag{158}$$

The result is that

$$\bar{\psi}\left(\frac{1}{z_0}\right) = C_1 + iC_2 = \frac{D_1 + iD_2}{(1 - 2\alpha^2 \cos 2\beta + \alpha^4)(1 - \alpha^2)} \tag{159}$$

and B_1 and B_2 are given by (142).

Since it is not possible to work out simple expressions for B_1 and B_2 in terms of the coefficients α, β, θ_1 and θ_2 we now work a numerical case. Let (143) be

$$a_n - 1.4x_{n-1} + 0.98x_{n-2} = a_n + a_{n-1} - a_{n-2} \tag{160}$$

In this case we have

$$\theta_1 = -1; \qquad \theta_2 = 1$$

$$\alpha^2 = 0.98; \qquad \beta = \frac{\pi}{4}$$

$$\cos\beta = \sin\beta = 1/\sqrt{2}$$

$$\alpha \sin\beta = \alpha\cos\beta = 0.7$$

$$\cos 2\beta = 0, \qquad \sin 2\beta = 1$$

substitution of the appropriate values in (148) gives

$$P(z_0) = 0.98571 + i(1.41429) \tag{161}$$

and using (136) we obtain

$$A_1 = 1.40815$$

$$A_2 = 2.020414 \tag{162}$$

From (146)

$$A_0 = -1.02041$$

So the impulse response (150) becomes

$$\psi_n = -1.02041\ \delta_n + (0.98)^{n/2}\left(1.40815 \sin\frac{n\pi}{4} + 2.02041 \cos\frac{n\pi}{4}\right) \tag{163}$$

We further have from (155)

$$\bar{\psi}\left(\frac{1}{z_0}\right) = \frac{[1 + 0.7(1 + i) - 0.98i](1 + 0.98i)}{(1 + 0.98^2)(1 - 0.98)}$$

$$= \frac{1.9744 + 1.386i}{0.039208} = 50.357 + 35.35i \tag{164}$$

and with (142), (159) and (162)

$$B_1 = -0.51143$$

$$B_2 = 151.52$$

It therefore follows that the function f_k is

$$f_k = -1.02041\ \delta_k + (0.98)^{k/2}\left(-0.51143 \sin\frac{k\pi}{4} + 151.52 \cos\frac{k\pi}{4}\right) \tag{165}$$

6. SUMMARY OF SOLVED CASES

This is a summary of the form of the impulse response and of the autocovariance function corresponding to the most frequently encountered ARMA equations. In listing the various cases we use the numbers (1), (2), (3) on the left margin to denote:

(1) The defining ARMA equation

(2) The impulse response ψ_n, and its coefficients

(3) The normalized autocovariance function g_k, and its coefficients

We note that the autocovariance function of the filter output x_n is g_k multiplied by the variance σ_u^2 of the filter input a_n. Also, the expression given for g_k represents the autocovariance function only for $k \geq 0$. For $k < 0$ the symmetry is used, that is $g_{-k} = g_k$.

<u>CASE 1</u>: ARMA (0, 1)

(1) $x_n = a_n - \theta_1 a_{n-1}$

(2) $\psi_n = \delta_n - \theta_1 \delta_{n-1}$

(3) $g_k = (1 + \theta_1^2)\delta_k - \theta_1 \delta_{k-1}$

CASE 2: ARMA (0, 2)

(1) $x_n = a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2}$

(2) $\psi_n = \delta_n - \theta_1 \delta_{n-1} - \theta_2 \delta_{n-2}$

(3) $g_k = (1+\theta_1^2+\theta_2^2)\delta_k + (-\theta_1+\theta_1\theta_2)\delta_{k-1} - \theta_2\delta_{k-2}$

CASE 3: ARMA (0, 3)

(1) $x_n = a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2} - \theta_3 a_{n-3}$

(2) $\psi_n = \delta_n - \theta_1 \delta_{n-1} - \theta_2 \delta_{n-2} - \theta_3 \delta_{n-3}$

(3) $g_k = (1+\theta_1^2+\theta_2^2+\theta_3^2)\delta_k + (-\theta_1+\theta_1\theta_2+\theta_2\theta_3)\delta_{k-1} + (-\theta_2+\theta_1\theta_3)\delta_{k-2} - \theta_3\delta_{k-3}$

CASE 4: ARMA (1, 0)

(1) $x_n - \phi_1 x_{n-1} = a_n$

(2) $\psi_n = \phi_1^n$

(3) $g_k = \phi_1^k/(1 - \phi_1^2)$

CASE 5: ARMA (1, 1)

(1) $x_n - \phi_1 x_{n-1} = a_n - \theta_1 a_{n-1}$

(2) $\psi_n = A_0\delta_n + (1-A_0)\phi_1^n; \quad A_0 = \theta_1/\phi_1$

(3) $g_k = B_0\delta_k + B_1\phi_1^k; \; B_0 = A_0; \; B_1 = \dfrac{(1 - \theta_1\phi_1)(1 - A_0)}{1 - \phi_1^2}$

CASE 6: ARMA (1, 2)

(1) $x_n - \phi_1 x_{n-1} = a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2}$

(2) $\psi_n = A_0\delta_n + A_1\delta_{n-1} + A_2\phi_1^n$

$$A_0 = \frac{\theta_1\phi_1 + \theta_2}{\phi_1^2}; \quad A_1 = \frac{\theta_2}{\phi_1}; \quad A_2 = 1 - A_0$$

(3) $g_k = B_0\delta_k + B_1\delta_{k-1} + B_2\phi_1^k$

$$B_0 = A_0 + (\phi_1 - \theta_1)A_1; \quad B_1 = A_1; \quad B_2 = \frac{1 - \theta_1\phi_1 - \theta_2\phi_1^2}{1 - \phi_1^2} A_2$$

CASE 7: ARMA (1, 3)

(1) $x_n - \phi_1 x_{n-1} = a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2} - \theta_3 a_{n-3}$

(2) $\psi_n = A_0\delta_n + A_1\delta_{n-1} + A_2\delta_{n-2} + A_3\phi_1^n$

$$A_0 = \frac{\theta_1\phi_1^2 + \theta_2\phi_1 + \theta_3}{\phi_1^3}; \quad A_1 = \frac{\theta_2\phi_1 + \theta_3}{\phi_1^2}; \quad A_2 = \frac{\theta_3}{\phi_1}; \quad A_3 = 1 - A_0$$

(3) $g_k = B_0\delta_k + B_1\delta_{k-1} + B_2\delta_{k-2} + B_3\phi_1^k$

$$B_0 = A_0 + (\phi_1 - \theta_1)A_1 + (\phi_1^2 - \theta_1\phi_1 - \theta_2)A_2; \quad B_1 = A_1 + (\phi_1 - \theta_1)A_2;$$

$$B_2 = A_2; \quad B_3 = \frac{1 - \theta_1\phi_1 - \theta_2\phi_1^2 - \theta_3\phi_1^3}{1 - \phi_1^2} A_3$$

CASE 8: ARMA (2, 0)

Set $\theta_1 = 0$ in Cases 9, or 10, or 13.

CASE 9: ARMA (2, 1) where the characteristic polynomial has two real zeros, That is, $z = \alpha$ and $z = \beta$. We then have

$$z^2 - \phi_1 z - \phi_2 = (z - \alpha)(z - \beta)$$

(1) $x_n - (\alpha + \beta)x_{n-1} + \alpha\beta x_{n-2} = a_n - \theta_1 a_{n-1}$

(2) $\psi_n = A_0\alpha^n + A_1\beta^n; \quad A_0 = \dfrac{\alpha - \theta_1}{\alpha - \beta}; \quad A_1 = \dfrac{\beta - \theta_1}{\beta - \alpha};$

(3) $g_k = B_0\alpha^k + B_1\beta^k; \quad B_0 = \dfrac{I - \alpha\theta_1}{(1 - \alpha^2)(1 - \alpha\beta)} A_0; \quad B_1 = \dfrac{1 - \beta\theta_1}{(1 - \alpha\beta)(1 - \beta^2)} A_1$

CASE 10: ARMA (2, 1) where the characteristic polynomial has a double zero at $z = \beta$. We then have

$$z^2 - \phi_1 z - \phi_2 = z^2 - 2\beta z + \beta^2$$

(1) $x_n - 2\beta x_{n-1} + \beta^2 x_{n-2} = a_n - \theta_1 a_{n-1}$

(2) $\psi_n = A_0 n\beta^{n-1} + A_1 \beta^n \; ; \; A_0 = \beta - \theta_1 \; ; \; A_1 = 1$

(3) $g_k = B_0 k\beta^{k-1} + B_1\beta^k \; ; \; B_0 = \dfrac{(\beta - \theta_1)(1 - \beta\theta_1)}{(1 - \beta^2)^2} \; ; \; B_1 = \dfrac{(1+\beta^2)(1+\theta_1^2) - 4\beta\theta_1}{(1 - \beta^2)^3}$

CASE 11: ARMA (2, 2) where the characteristic polynomial has two real zeros, that is, $z = \alpha$ and $z = \beta$. We then have

$$z^2 - \phi_1 z - \phi_2 = (z - \alpha)(z - \beta)$$

(1) $x_n - (\alpha + \beta)x_{n-1} + \alpha\beta x_{n-2} = a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2}$

(2) $\psi_n = A_0\delta_n + A_1\alpha^n + A_2\beta^n$

$$A_0 = -\frac{\theta_2}{\alpha\beta} \; ; \; A_1 = \frac{\alpha^2 - \alpha\theta_1 - \theta_2}{\alpha(\alpha - \beta)} \; ; \; A_2 = \frac{\beta^2 - \beta\theta_1 - \theta_2}{\beta(\beta - \alpha)}$$

(3) $g_k = B_0\delta_k + B_1\alpha^k + B_2\beta^k$

$$B_0 = A_0 \; ; \; B_1 = \frac{1 - \alpha\theta_1 - \alpha^2\theta_2}{(1 - \alpha^2)(1 - \alpha\beta)} A_1 \; ; \; B_2 = \frac{1 - \beta\theta_1 - \beta^2\theta_2}{(1 - \alpha\beta)(1 - \beta^2)} A_2$$

CASE 12: ARMA (2, 3) where the characteristic polynomial has two real zeros, that is, $z = \alpha$ and $z = \beta$. We then have

$$z^2 - \phi_1 z - \phi_z = (z - \alpha)(z - \beta)$$

(1) $x_n - (\alpha + \beta)x_{n-1} + \alpha\beta x_{n-2} = a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2} - \theta_3 a_{n-3}$

(2) $\psi_n = A_0\delta_n + A_1\delta_{n-1} + A_2\alpha^n + A_3\beta^n$

$$A_0 = -\frac{\theta_2}{\alpha\beta} - \frac{(\alpha + \beta)\theta_3}{(\alpha\beta)^2} \; ; \; A_1 = -\frac{\theta_3}{\alpha\beta} \; ; \; A_2 = \frac{\alpha^3 - \alpha^2\theta_1 - \alpha\theta_2 - \theta_3}{\alpha^2(\alpha - \beta)} \; ;$$

$$A_3 = \frac{\beta^3 - \beta^2\theta_1 - \beta\theta_2 - \theta_3}{\beta^2(\beta - \alpha)}$$

(3) $g_k = B_0\delta_k + B_1\delta_{k-1} + B_2\alpha^k + B_3\beta^k$

$$B_0 = A_0 + (\alpha + \beta - \theta_1)A_1 \quad ; \quad B_1 = A_1$$

$$B_2 = \frac{1 - \alpha\theta_1 - \alpha^2\theta_2 - \alpha^3\theta_3}{(1 - \alpha^2)(1 - \alpha\beta)} \quad ; \quad B_3 = \frac{1 - \beta\theta_1 - \beta^2\theta_2 - \beta^3\theta_3}{(1 - \alpha\beta)(1 - \beta^2)} \; A_3$$

<u>CASE 13</u>: ARMA (2, 1) where the characteristic polynomial has complex conjugate zeros, that is, $z_0 = \alpha e^{i\beta}$ and $z_1 = \alpha e^{-i\beta}$. We then have

$$z^2 - \phi_1 z - \phi_2 = z^2 - 2\alpha z \cos\beta + \alpha^2$$

(1) $x_n - 2\alpha(\cos\beta)x_{n-1} + \alpha^2 x_{n-2} = a_n - \theta_1 a_{n-1}$

(2) $\psi_n = \alpha^n(A_0 \sin n\beta + A_1 \cos n\beta)$

$$A_0 = \cot\beta - \frac{\theta_1}{\alpha \sin\beta} \; ; \quad A_1 = 1$$

(3) $g_k = \alpha^k(B_0 \sin k\beta + B_1 \cos k\beta)$

$$B_0 = A_0C_1 - A_1C_2 \; ; \quad B_1 = A_0C_2 + A_1C_1$$

$$C_1 = \mathrm{Re}\; \bar{\psi}\left(\frac{1}{z_0}\right) = \frac{1 + \alpha^4 + \theta_1(\alpha^3 - \alpha)\cos\beta - \alpha^2 \cos 2\beta}{(1 - 2\alpha^2 \cos 2\beta + \alpha^4)(1 - \alpha^2)}$$

$$C_2 = \mathrm{Im}\; \bar{\psi}\left(\frac{1}{z_0}\right) = \frac{-\theta_1(\alpha + \alpha^3)\sin\beta + \alpha^2 \sin 2\beta}{(1 - 2\alpha^2 \cos 2\beta + \alpha^4)(1 - \alpha^2)}$$

<u>CASE 14</u>: ARMA (2, 2) where the characteristic polynomial has complex conjugate zeros, that is, $z_0 = \alpha e^{i\beta}$ and $z_1 = \alpha e^{-i\beta}$. We then have

$$z^2 - \phi_1 z - \phi_2 = z^2 - 2\alpha z \cos\beta + \alpha^2$$

(1) $x_n - 2\alpha(\cos\beta)x_{n-1} + \alpha^2 x_{n-2} = a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2}$

(2) $\psi_n = A_0\delta_n + \alpha^n(A_1 \sin n\beta + A_2 \cos n\beta)$

$$A_0 = -\frac{\theta_2}{\alpha^2}\ ;\quad A_1 = (1 - \frac{\theta_2}{\alpha^2})\cot\beta - \frac{\theta_1}{\sin\beta}\ ;\quad A_2 = (1 + \frac{\theta_2}{\alpha^2})$$

(3) $g_k = B_0\delta_k + \alpha^k(B_1 \sin k\beta + B_2 \cos k\beta)$

$$B_0 = A_0\ ;\quad B_1 = A_1C_1 - A_2C_2\ ;\quad B_2 = A_1C_2 + A_2C_1$$

$$C_1 = \mathrm{Re}\ \bar{\psi}(\frac{1}{z_0}) = \frac{1 + \theta_2\alpha^4 + \theta_1(\alpha^3 - \alpha)\cos\beta - \alpha^2(1 + \theta_2)\cos 2\beta}{(1 - 2\alpha^2 \cos 2\beta + \alpha^4)(1 - \alpha^2)}$$

$$C_2 = \mathrm{Im}\ \bar{\psi}(\frac{1}{z_0}) = \frac{-\theta_1(\alpha + \alpha^3)\sin\beta + \alpha^2(1 - \theta_2)\sin 2\beta}{(1 - 2\alpha^2 \cos 2\beta + \alpha^4)(1 - \alpha^2)}$$

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